Application of Internal Gravitational Field Equations to Geophysical Measurement of G

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We present the spherical harmonic equation for the gravitational potential inside the Earth. The equation satisfies Poisson’s equation and converges uniformly. It obviates the need for downward continuation of the exterior potential with its attendant convergence difficulties but of course requires some knowledge of the Earth’s density distribution. Equipped with the equation, we derive the general expression for the geophysical measurement of the gravitational constant G made inside the Earth, such as in boreholes and mine shafts. We also present numerical evidence that the long- to intermediate-wavelength gravity anomalies can masquerade as the “fifth force” if not properly corrected for.

INTRODUCTION

The spherical harmonic expansion for the gravitational potential outside the Earth is well-known and given in standard textbooks (e.g., Heiskanen and Moritz [1967, p. 59], Kaula [1968, p. 67], Jeffreys [1976], and many others). However, what is not given is the corresponding expression for the potential inside the Earth. Instead, the equation for the exterior potential is often downward continued inside the Earth, which can lead to convergence difficulties.

We provide the spherical harmonic expansion for the gravitational potential inside the Earth: the equation satisfies Poisson’s equation and converges uniformly. We then use the equation to comment on geophysical measurements of the gravitational constant G made inside the Earth (such as in boreholes and mine shafts) used as tests for non-Newtonian gravity. We close with a discussion about the free-air gravity anomaly in these experiments and present numerical evidence that the long- to intermediate-wavelength gravity anomalies can masquerade as the “fifth force” if not properly corrected for.

THE EQUATION

Let us fix the origin of our coordinate system at the Earth’s center of mass. In terms of spherical harmonics, the mass density $\rho(r)$ at position $r$ can be written as

$$\rho(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\Omega)$$

(1)

where $\rho_{lm}(r)$ is the harmonic expansion coefficient which depends only on radial distance $r$. The spherical harmonic functions can be expressed as

$$Y_{lm}(\Omega) = P_{lm}(\cos \theta) \cos m\lambda$$

(2a)

$$Y_{lm}(\Omega) = P_{lm}(\cos \theta) \sin m\lambda$$

(2b)

where $P_{lm}$ is the normalized associated Legendre function of degree $l$ and order $m$, and $i$ specifies either the cosine or sine term. The position vector is $r = (r, \Omega)$, where $\Omega$ is the solid angle representing colatitude $\theta$ and longitude $\lambda$. We adopt Kaula’s [1967] $4\pi$ normalization over the unit sphere.

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$$\int Y_{lm}(\Omega) Y_{lm'}(\Omega) \ d\Omega = 4\pi \delta_{ll'} \delta_{mm'}$$

(3)

Let $R$ be a radius enclosing all of the Earth. The gravitational potential at an internal point ($r < R$) is

$$U(r) = 4\pi G \sum_{lm} \frac{1}{2l+1} Y_{lm}(\Omega)$$

$$\cdot \left[ \int_0^r \rho_{lm}(s) \left( \frac{s^l+2}{s^{l+1}} \right) ds + \int_r^R \rho_{lm}(s) \left( \frac{r^l}{s^{l-1}} \right) ds \right]$$

(4)

where $G$ is the universal gravitational constant, and the summations over $l$, $m$, and $i$ are the same as (1). This general form is not given in standard geodesy texts (with perhaps the exception of a related short discussion in the work by Hotine [1969, p. 173]; but see Petrovskaya [1977]).

Equation (4) can be readily derived from considerations of the sphere $s < r$ and the shell $r < s < R$ as is regularly done in electrostatic field theory. Here, we prove that the above expression satisfies Poisson’s equation. We rewrite the potential

$$U(r) = \sum_{lm} U_{lm}(r) = \sum_{lm} u_{lm}(r) Y_{lm}(\Omega)$$

where

$$u_{lm}(r) = \frac{4\pi G}{2l+1} \left[ \int_0^r \rho_{lm}(s) \left( \frac{s^l+2}{s^{l+1}} \right) ds + \int_r^R \rho_{lm}(s) \left( \frac{r^l}{s^{l-1}} \right) ds \right]$$

(5)

with the first integral being due to the sphere and the second due to the shell. Then for each term

$$\nabla^2 U_{lm}(r) = (l(l+1)\rho^2 \delta_{lm}) \nabla^2 Y_{lm}(\Omega)$$

$$- 4\pi G (l(l+1)\rho^2) u_{lm}(r) Y_{lm}(\Omega)$$

(6)

where $\rho = \partial \rho / \partial r$ [e.g., Morse and Feshbach, 1953, p. 1264]. By the rule for differentiating integrals [e.g., Rudin, 1964, p. 114], we get

$$\partial_r \left[ r^l \partial_r u_{lm}(r) \right] = 4\pi G (l+1) u_{lm}(r) - 4\pi G r^2 \rho_{lm}(r)$$

(7)

Combining (6) with (7) and summing over $l$, $m$, $i$ gives
\[ \nabla^2 U(r) = -4\pi G \sum_{lm} \rho_{lm}(r) = -4\pi G \rho(r) \]  

(8)

which is Poisson's equation.

We will now show that (4) is well-behaved inside the Earth as the maximum degree for the summation over \( l \) approaches infinity, under mild mathematical restrictions on the density. We write (1) as \( \rho(s) = \sum_s f_i \), where \( s \) is the position vector, and

\[ f_i = \sum_{m=0}^{2} \sum_{l=1}^{\infty} \rho_{ml}(s) Y_{lm}(\Omega) \]  

(9)

Assume that (9) is integrable and has uniformly bounded sums. Also, set

\[ W_l = r/(2l+1)(s/r) \]  

(10)

Then \( W_l \rightarrow 0 \) uniformly as \( l \rightarrow \infty \), and \( W_l(s) \geq W_l(s) \geq \ldots \) for all \( 0 \leq s \leq r \). Therefore \( \sum_w f_i W_l \) converges uniformly \[ [Rudin, 1964, p. 155]\]. The integral of this sum exists and yields the first term of sums in (4) (apart from a constant factor \( 4\pi G \)), after interchanging the order of summation and integration, as is permissible for uniform convergence \[ [Rudin, 1964, p. 138]\]. By similar reasoning the second set of sums in (4) can be shown to be convergent. Therefore the sum of the two sets is convergent, which is (4).

**APPLICATION TO MEASUREMENT OF G**

We now apply (4) to geophysical experiments inside the Earth which test for non-Newtonian gravity. Geophysical determinations of \( G \) over distances too large for laboratory measurement have resulted in values on the order of a few tenths of 1% larger than the laboratory value \[ [Stacey and Tuck, 1981; Stacey et al., 1981, 1988; Stacey, 1983; Holding and Tuck, 1984; Holding et al., 1986; Hsui, 1987]\]. This has been interpreted as tentative evidence that gravity is non-Newtonian.

These \( G \) values are derived from measurements of the gravitational acceleration \( g \) conducted in mine shafts, boreholes, and ocean depths, together with a (Newtonian) theory of how \( g \) varies inside the Earth. Briefly, the technique is to assume a rotationally flattened Earth and to remove ellipsoidal shells above the depth at which \( g \) is measured. Since the gravitational attraction is zero inside a homogeneous ellipsoidal shell, only the ellipsoid below the gravimeter contributes to \( g \) \[ cf. Dahlen, 1982\]. By correcting for the mass of the shells, one knows how much mass remains within the ellipsoid, and therefore its pull upon the gravimeter. Departures from the laboratory \( G \) value of \( 6.6726(5) \times 10^{-11} \) \( \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \) \[ Luther and Towler, 1982\] at varying depths inside the Earth provide the evidence for non-Newtonian gravity.

It is the local density, however, that appears in the equations of the theory, and densities in the local environment are carefully measured. The local density is not usually characteristic of the average density of the shell which must be used in computing its mass. For example, the local density of a borehole on a continent is about \( 2800 \) \( \text{kg m}^{-3} \), while at a depth just below sea level the average density will be approximately \( (0.7) \times 1000 + (0.3) \times 2800 \approx 1540 \) \( \text{kg m}^{-3} \), reflecting the various proportions of water and continental rock. This example illustrates that the average density can be much different from the local density. This "paradox" will be resolved below.

We ignore the Earth's rotation and the resultant hydrostatic flattening in the Earth, since they are easily corrected for. We write the first integral in (4) as \( f_0^R - f_0^R \), and (4) becomes

\[ U(r) = \frac{GM}{R} \sum_{lm} C_{lm} Y_{lm}(\Omega) \left( \frac{R}{r} \right)^{l+1} + 4\pi G \sum_{lm} \frac{1}{2l+1} Y_{lm}(\Omega) \]

\[ \cdot \int_{r}^{R} \rho_{lm}(s) \left( \frac{s^{l+2} + r^{l}}{s^{l+1} + r^{l}} \right) ds \]  

(11)

where \( M \) is the mass of the Earth, and the terms in the integral with the minus sign together with the first sum give back the potential due to the sphere \( s < r \). The \( C_{lm} \) are the normalized multipoles of density \( \rho(r) \), given by

\[ C_{lm} = \frac{1}{(2l+1)M} \int_{V} \rho(s) Y_{lm}(\Omega) r^{l} dv \]  

(12)

where \( V \) is the volume of the Earth. They are the usual harmonic coefficients (often known as the Stokes coefficients) of the external \( (r > R) \) gravitational field \[ e.g., Kaula, 1968\]:

\[ U_{\text{rad}}(r) = \frac{GM}{R} \sum_{lm} C_{lm} Y_{lm}(\Omega) \left( \frac{R}{r} \right)^{l+1} \]  

(13)

At first sight (11) appears to diverge, since \( r < R \) in the first sum. However, terms in the second sum cancel those in the first so that the equation converges.

Differentiating (11) with respect to \( r \) gives the following expression for the magnitude of the gravitational acceleration:

\[ g(r) = -\partial_r U(r) = GM \sum_{lm} Y_{lm}(\Omega) \left( \frac{R}{r} \right)^{l+1} \]

\[ - 4\pi G \sum_{lm} \frac{1}{2l+1} Y_{lm}(\Omega) \]

\[ \cdot \int_{r}^{R} \rho_{lm}(s) \left[ \left( \frac{l+1}{r} \right)^{l+2} + \left( \frac{s}{r} \right)^{l+1} - l(l-1) \right] ds \]  

(14)

The slight difference in the local vertical and radial directions has been neglected.

Now in order to compare with previous studies (see references above) which measured the radial gradient of \( g(r) \), we differentiate (14) with respect to \( r \) and write

\[ -\partial_r g(r) = (g_0/R) \hat{Q}(r) - 4\pi G \hat{I}(r) + \hat{P}(r) \]  

(15)

where \( g_0 = GM/R^2 \), and

\[ \hat{Q}(r) = \sum_{lm} (l+1)(l+2) C_{lm} Y_{lm}(\Omega) \left( \frac{R}{r} \right)^{l+3} \]

\[ \hat{P}(r) = \sum_{lm} Y_{lm}(\Omega) \]

\[ \cdot \int_{r}^{R} \rho_{lm}(s) \left[ \left( l+1 \right) \left( l+2 \right) \left( \frac{s}{r} \right)^{l+2} - l(l-1) \left( \frac{r}{s} \right)^{l+1} \right] ds \]  

(16)

(17)

Let us now examine (15)-(17). Suppose a gravity gradient measurement is made at position \( r \) inside the Earth \( (r < R) \).
The local density $\rho(r)$, and not the average layer density, appears on the right side of (15). Hence it is, indeed, the local density which must be used when integrating (15) to find the variation in $g(r)$ between two depths, as done by Stacey and Tuck [1981], for example. This resolves the above-mentioned "paradox." Note also that the effect of the density anomalies in the shell above the experiment location is expressed in the quantity $\tilde{P}$. The density anomalies inside the sphere of radius $r$ are absorbed in the $C_{i m},$ coefficients, which are for the entire planet, in the quantity $\tilde{Q}$.

Rearranging (15) and writing $\tilde{P}(r) = P_0(r) + P(r)$ and $\tilde{Q}(r) = Q_0(r) + Q(r)$, where subscript 0 indicates the $l = 0$ (monopole) terms and no subscript indicates the sum of all higher-degree terms $l \geq 2$ (the $l = 1$ terms vanish because the origin coincides with the center of mass), we have

$$G \frac{1}{4\pi[\rho(r) + P_0(r) + P(r)]} \cdot (\delta, g(r) + g_0/R[Q_0(r) + Q(r)])$$

(18)

This equation gives a quantitative geophysical means for testing Newtonian gravity by measuring all quantities on the right side and computing $G$. If it does agree with the laboratory value within experimental error, then this is evidence that the gravity anomalies are the source of the reported non-Newtonian behavior. Equation (18) reduces, when $P(r)$ and $Q(r)$ are set to zero, to the formula given in previous studies (which, apart from corrections for hydrostatic equilibrium and rotation, considered only the $P_0(r)$ and $Q_0(r)$ terms in their solution for $G$). Thus it constitutes a generalization of (6) of Stacey et al. [1981], for instance, to include $P(r)$ and $Q(r)$ which arise from density anomalies. In the next section we shall use it in a specialized model that concentrates on long to intermediate wavelength anomalies.

**RESULTS AND DISCUSSION**

Let us examine (18) for the geophysical determination of $G$. Although $\tilde{P}$ and $\tilde{Q}$ involve global integrals, it is obviously the local density anomalies that will have the largest influence. Hsu [1987], for instance, plots $g$ as a function of depth in a borehole; the graph shows a break due to a nearby small reef formation. The local density anomalies for the shell integral $\tilde{P}$ can in principle be measured and taken into account despite practical difficulties, as discussed by Stacey et al. [1981, 1988]. D. Bartlett and W. Tew (The Hill in North Carolina and the sixth force, submitted to Physical Review Letters, 1989) also presents evidence that local topography can have significant effects. Correcting for that, only one non-Newtonian force (instead of two as previously reported by Eckhardt et al. [1988a]; see also Romanides et al. [1989]) is required to fit the observed data in the tower experiment [Eckhardt et al., 1988b].

To correct for the local density anomalies that appear in $\tilde{Q}$ is more problematic. Since $g_0Q(r)/R$ is basically the free-air gravity gradient, we here enquire as to how the latter has been treated in previous studies. Holding and Tuck [1984] caution that their result may be biased by regional gravity anomalies. Holding et al. [1986] discuss the free-air gradient with inconclusive results, while Hsu [1987] does not mention the free-air gradient. Stacey and Tuck [1981] correct for a reference gradient but do not state a value. Stacey et al. [1981], do quote a free-air gradient discrepancy of $8 \times 10^{-5}$ s$^{-2}$, but absorb it into their error budget rather than correct for it. Had they corrected for it, according to (19), they would have gotten a $G$ value of $6.47 \times 10^{-11}$ instead of $6.71 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$. That is 3% lower than the laboratory value, so that the fifth force would have changed sign. Moreover, Parker [1988] and Kim [1989] have demonstrated the need to consider possible unmodeled local density anomalies; these density anomalies, if properly situated, can account for the reported deviations in the geophysically measured $G$ so that no non-Newtonian explanation is required. It should be mentioned that the free-air anomaly can alternatively be found from Poisson's integral [Heiskanen and Moritz, 1967, p. 89]. This is the approach of Eckhardt et al. [1988a] in the tower experiment, since the tower is outside the Earth. In principle, it takes into account the complete free-air anomaly, from all sources distant and local. Using this approach, Stacey et al. [1988] discuss the uncertainty arising from the free-air anomalies by examining a smoothed representation of gravity survey data out to 200 km from the experiment site. They report an estimated free-air correction whose sign is consistent with a non-Newtonian explanation.

Our purpose here is to investigate the complementary part of the wavelength spectrum. Specifically, we address whether the long to intermediate wavelengths of the Earth's gravitational field due to global density anomalies can significantly affect the geophysically determined $G$. We do this by examining their contributions to $P(r)$ and $Q(r)$ in (18). For that purpose, we consider the case of a hypothetical planet with the mass and radius of the Earth (but not the rotation and the flattening, see above) whose coefficients (12) are zero for $l > 180$. The scale length under consideration is thus greater than about 100 km.

We consider $\tilde{P}(r)$ first and show that it can be neglected in our model. If we assume the continents on our hypothetical planet to have constant densities and vertical boundaries for the depths considered, then $\rho_{i m}(r) = \text{const}$ and can be moved outside the integral in (17). If $l_z \ll R$, where $z = R - r$ is the depth to the point where $g$ is measured, then this integral becomes approximately $2l/(2l + 1)z$, so that $P(r) = 2z/RP(r)$. For typical depths (e.g., in a borehole) of 1 km, $\tilde{P}(r) = 0.0003 \rho(r)$ and hence can be neglected in (15), at least for the present model.

Therefore the effect of the long- and intermediate-wavelength gravity anomalies on the solution of $G$, as a function of the geographical location, becomes

$$\Delta G(r) = g_0Q(r)/(4\pi R P(r))$$

(19)

Note that only the multipoles of the density distribution (the $C_{i m}$ coefficients) are involved; the complete knowledge of the density itself is not necessary.

Our task now is to evaluate $Q(r)$. For depths $z \ll R$ and the long to intermediate wavelength gravity anomalies we are considering ($2 \leq l \leq 180$), the factor $(R/r)^{l+3}$ in (16) can be taken to be 1 with an error of $(l + 3/zR)$ by a Taylor series expansion. Thus for depths of ~1 km, (16) may be evaluated at the planet's surface with only about a few percent error in $Q(r)$. Since the correction (19) is itself a small quantity, the error made in evaluating $Q(r)$ at the surface will yield only higher-order corrections in $\Delta G(r)$ and may be safely neglected.

For the $C_{i m}$ coefficients, $2 \leq l \leq 180$, we use those of the GEM 10C gravitational field [Lerch et al., 1981], where the $C_{201}$ and $C_{401}$ coefficients are corrected for hydrostatic flattening [Nakiboglu, 1979]. Using the resulting $Q(r)$ in (19)
gives $\sigma_G / G \approx 0.14\%$, where $\sigma_G$ is the root-mean-square of $\Delta G(r)$ over the Earth's surface, with $\Delta G(r)$ being computed at the center of $1^\circ \times 1^\circ$ squares. The computed $\Delta G(r)$ for our hypothetical planet thus in average approaches the range of the observed deviations cited for non-Newtonian gravity.

The above is for $L = 180$, $L$ being the cut-off harmonic degree for the summation over $l$. Let us now examine the dependence of $\Delta G$ on $L$. We have computed $\sigma_G$ values as a function of $L$ for $L = 36, 72, 108, 144$ as well as $L = 180$. We find that $\sigma_G$ grows approximately as $\sigma_G \sim L^{-1.2}$ in this range, consistent with Kaula's rule-of-thumb [Kaula, 1967]. This empirical rule states that the $C_{l,m}$ coefficients decrease as $\pm 10^{-1} l^{-2}$, which leads to a growth of $\sigma_G$ in proportion to $L$ based on a random walk statistic. Kaula's rule only begins to fail past $l \approx 200$, where the coefficients decrease at a slightly faster rate [Rapp and Cruz, 1986]. At higher degrees ($1 \sim 1000$) the degree variances decay at a rate $l^{-6}$ faster than Kaula's rule, at least for data taken over Canada [Vassiliou and Schwarz, 1987]. Hence we expect larger $\Delta G(r)$ values for maximum degrees greater than the $L = 180$ considered here.

**Summary**

We have derived a spherical harmonic expression (equation (15)) for the gravitational potential inside the Earth. It formed the basis for our study on the importance of large-scale gravity anomalies on measurements of $G$ made inside the Earth. We did not set up (15) as a general new method in non-Newtonian gravity studies, nor did we attempt to provide corrections for specific sites. Rather, our more modest goal was to determine whether the long-to-intermediate-wavelength gravity anomalies of the global field can affect the measurement of $G$ at the level reported for non-Newtonian gravity. To obtain the answer to this question we constructed a specialized Earth model whose gravity field is the nonhydrostatic part of the GEM-10C gravity field. Since this field has only a limited number of spherical harmonic coefficients (up to degree 180), all of the series appearing in our equations converge. We find that, for this model, the correction $\Delta G(r) = P_0(r) + P(r)$ in (18) is unimportant for the wavelengths considered. However, we did find that the free-air gradient embodied in $Q(r)$ in (18) can affect the measurement of $G$ at the level reported for non-Newtonian gravity. This emerged from a statistical study.

This result led us to enquire as to whether the free-air gravity gradient can be responsible for at least some of the reported non-Newtonian results. A survey of the literature indicated that this is indeed a possibility. We therefore urge caution in interpreting $G$ measurements made inside the Earth as evidence for non-Newtonian gravity until the gravity anomaly problem is resolved.

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