The external gravitational field produced by a rigid body of uniform density but irregular shape is formulated in terms of spherical harmonics. The formalism is applied to the Martian satellite Phobos. Based on a 3-dimensional shape model of Phobos by Duxbury, the gravitational coefficients up to degree and order 4 for a homogeneous Phobos are computed. In particular, we find $J_2 = 0.105$. The in-plane libration amplitude of a homogeneous Phobos is predicted to be $0.97^\circ$, within the rather large uncertainty of the observed value of $0.78 \pm 0.4^\circ$.

Abstract. The external gravitational field produced by a rigid body of uniform density but irregular shape is formulated in terms of spherical harmonics. The formalism is applied to the Martian satellite Phobos. Based on a 3-dimensional shape model of Phobos by Duxbury, the gravitational coefficients up to degree and order 4 for a homogeneous Phobos are computed. In particular, we find $J_2 = 0.105$. The in-plane libration amplitude of a homogeneous Phobos is predicted to be $0.97^\circ$, within the rather large uncertainty of the observed value of $0.78 \pm 0.4^\circ$.

Introduction

It is well known that the external gravitational field of a variable-density planet under isostatic equilibrium (e.g., the Earth) carries little signature of the planet's surface topography. The present paper deals with the opposite situation in which the body under consideration is assumed to be homogeneous, and hence of uniform density, so that the gravitational field is completely determined by its topography, or shape.

The impetus is the irregularly shaped Phobos, one of the two natural satellites of Mars. Its 3-dimensional shape, roughly 27 km x 22 km x 18 km, has been modeled in some detail based on Viking images [Turner, 1978; Wu et al., 1988; Duxbury and Callahan, 1989; Duxbury, 1989]. Whether Phobos is homogeneous bears crucially on models of the origin and evolution of the Martian satellites [e.g., Veverka and Burns, 1980; Thomas et al., 1986]. The gravitational field and its dynamical consequences that we shall compute for the homogeneous Phobos will serve as the "baseline" values, in the sense that their departures from observations can be used to infer Phobos' true density distribution.

It should be pointed out that the low-degree gravitational coefficients of a homogeneous Phobos have previously been calculated and used in dynamical analysis by various authors. These results, unfortunately, are in complete discord. For example, when subject to the same normalization, Bursa [1988] gave $C_{20} = -0.0350$ and $C_{22} = 0.213$ [see also Barkin, 1985; Chapront-Touze, 1988]. As far as we can determine (see below), this discrepancy is attributable to unsatisfactory theories as well as to differences in the shape models and in the coordinate setting adopted by different authors. The present paper remedies this by providing a clearly defined set of gravitational coefficients for Phobos, up to degree and order 4.

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Formulation

Let us first consider a body of arbitrary density $\rho(r)$. We define a phobocentric Cartesian coordinate system such that the x axis points to the mean sub-Mars point (on 0° meridian), the z axis coincides with the (mean) rotation axis, and the y axis points to the 90°E meridian to complete a right-handed system. The gravitational potential $U$ outside the body satisfies Laplace's equation, and can be expressed in terms of spherical harmonics [Kaula 1967]:

$$U(E_1) = \left( \frac{GM_0}{r_1} \right) \cdot \left\{ 1 + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left( \frac{\sigma}{r_1} \right)^l C_{l,m}(\Omega_1) \right\}$$

where $G$ is the gravitational constant, $M_0$ is the total mass of the body, $\sigma$ is the radius of the origin-centered sphere enclosing the body, $\Omega$ is the solid angle representing the colatitude $\theta$ and the longitude $\lambda$, and $(r_1, \theta_1, \lambda_1) = (r_1, \Omega_1)$ locates the field point $r_1$. $Y_{l,m}(\Omega_1)$ is the normalized spherical harmonic function of degree $l$ and order $m$, with the index $i$ designating the cosine or sine term in $\lambda$:

$$Y_{l,m}(\Omega_1) = P_{l,m}(\cos \Omega_1) \cos \lambda_1$$

$$Y_{l,m}(\Omega_1) = P_{l,m}(\cos \Omega_1) \sin \lambda_1$$

where $P_{l,m}$ is the normalized associated Legendre function satisfying the 4$\pi$ normalization of Kaula [1967] over the unit sphere:

$$\int Y_{l,m}(\Omega_1) Y_{l',m'}(\Omega_1) d\Omega = 4\pi \delta_{l1} \delta_{m1} \delta_{l'1} \delta_{m'1}$$

The dimensionless, normalized, real-valued gravitational harmonic coefficients $C_{l,m}$ specify the anomalous field completely. These triply indexed coefficients are related to the usual Stokes coefficients with $C_{l,m} \equiv C_{l,m}^m$ and $C_{l,m} \equiv S_{l,m}^m$, while the usual zonal $J^m$ coefficients of degree $l$ are given by $J^m \equiv (2l+1)^{1/2} C_{l,0}$. To obtain the un-normalized coefficients, one multiplies the factor $[(2 \delta_{l0})(2l+1)(1+m)]^{1/2}$ to $C_{l,m}$.

Comparing equation (1) with the multipole expansion of $U(r_1)$ [e.g., Jackson, 1975; for details see, e.g., Chao and Gross, 1987] one gets

$$C_{l,m} = \frac{1}{(2l+1) M_0 c^l} \int \rho(r_1) r_1 Y_{l,m}(\Omega_1) dV$$

where the integration is over the volume $V$ of the body. Thus the gravitational coefficients are in fact normalized multipoles of $\rho(r)$, and hence functionals of the shape of the body via (4). Our task now is to investigate this functional relationship in the case of a uniform density:

$$\rho(r) = \rho_v \quad \text{and} \quad M_0 = \rho V$$
Assumption (5) will remain invoked henceforth. Let \( R(\Omega) \) be the outer boundary of the body. Here we make the stipulation that the body in question is singly-connected and the origin is such that the function \( R(\Omega) \) is single-valued. This puts certain geometrical constraints on the body, to which small solar-system bodies do appear to conform. Substituting (5) and \( dV = r^2 dr d\Omega \) into (4) and performing the radial integral \( \int R(\Omega) \), one gets

\[
C_{l_m_l} = \frac{1}{1+3} \left( \frac{1}{21+1} \right) R(\Omega)^{1/3} \frac{1}{V^{1/2}}
\]

This surface integral is the basis for computing the gravitational field from the shape of a homogeneous body. Note that (6) is defined with respect to the reference radius \( \sigma \).

**Relation Between Harmonic Coefficients**

Suppose we expand the boundary surface of the body also in spherical harmonics:

\[
R(\Omega) = \frac{R^2}{1 + \Sigma \Sigma A_{l_m l} Y_{l_m l}(\Omega)}
\]

where \( R \) is a mean radius of the body defined by \( R = \int R(\Omega) d\Omega/4\pi \). The set of the normalized, real-valued harmonic coefficients \( A_{l_m l} \) will be referred to as the shape coefficients, alternatively expressed as \( A_{l_m l} \) and \( A_{l_m k} \), following Duxbury [1989]. The latter study has determined these coefficients for Phobos to harmonic degree and order 6, with \( R = 10.970 \) km.

We now wish to derive an analytical relationship which gives the gravitational coefficients in terms of the shape coefficients. To this end, we substitute equation (7) into (6) and obtain

\[
C_{l_m l} = \frac{3^{1/3}}{21+1} A_{l_m l}
\]

where we have invoked the orthogonality (4) and used short-hand notation \( \Sigma \Sigma \), etc., for triple summations. Obviously this relation is highly non-linear and entails a great deal of coupling among the shape coefficients. In light of this, the question arises: how practical is this procedure in obtaining the gravitational field? It is instructive to examine some simplified cases. First, consider the case where the shape is nearly spherical, so that \( |A_{l_m l}| \ll 1 \). For convenience we will choose to evaluate the gravitational coefficients at the nominal \( \sigma = R \), and use the approximation \( V = 4\pi R^2/3 \). Neglecting higher-order terms equation (8) reduces to the simple one-to-one relation:

\[
C_{l_m l} = \frac{3}{21+1} A_{l_m l}
\]

For \( l = 1 \), the two sides are identical. This is because the center of shape coincides to first order with the center of mass. For increasing \( l \), the gravitational coefficients become progressively smaller than the shape coefficients; the physical reason can be traced back to the inverse-square nature of the gravitational force.

For a body whose shape coefficients are not much smaller than unity, the first-order approximation (9) may yield rather unrealistic results. As an example, let us consider a spheroidal "Phobos" for which the only non-zero shape coefficient is \( A_{20} = -0.0862 \). The latter is converted from the unnormalized value determined by Duxbury [1989] for Phobos. What will be the corresponding gravitational coefficients? By virtue of a set of selection rules (derivable from simple symmetry considerations) for the integrals in (8), it can be shown that \( C_{l_m l} \) will vanish unless \( l \) is even, \( m = 0 \), and \( i = 1 \). Thus, the only gravitational coefficients that receive non-zero contributions from \( A_{20} \) are those of the even-degree zonal harmonics. Among these, only \( C_{20} \) receives first-order contribution from \( A_{20} \); but the second-order contribution is as large as 11% of the first-order contribution, and the third-order about 3%. For higher degree terms the deficiency of (9) is much more pronounced. The high order contribution \( C_{60} \) receives from \( A_{20} \) can be shown to be more than twice the first-order contribution it receives directly from \( A_{20} \). Similarly, those received by \( C_{60} \) from \( A_{20} \) is some 80% of the direct first-order contribution from \( A_{20} \).

We conclude that the first-order approximation (9), as used by Duxbury [1989], fails to give a realistic gravitational field for Phobos. If the shape coefficients are to be utilized for this purpose, the full equation (8) should be invoked. However, the evaluation of (8) is extremely tedious when the entire set of the shape coefficients is involved. Indeed, it proves more practical to compute the gravitational field by way of an alternative, yet more direct procedure, namely the numerical integration embodied in equation (6). We do this in the next section.

**Results and Discussion**

The numerical integration of equation (6) only requires the knowledge of surface boundary \( R(\Omega) \). Here we obtain Phobos' \( R(\Omega) \) from equation (7) based on the shape coefficients given by Duxbury [1989]. Using a surface grid of 5° x 5°, the volume of Phobos is obtained via a numerical integration over \( R(\Omega) \): \( V = (1/3) \int R^2(\Omega) d\Omega = 5756 \) km³. The gravitational coefficients, complete to degree and order 4 are then computed. For convenience, the reference radius is here stipulated to be the equivalent-volume radius: \( R_0 = (3V / 4\pi)^{1/3} \approx 11.12 \) km.

A discussion on the choice of the coordinate system is in order. The dipole \( l=1 \) shape coefficients, \( A_{10} \) and \( B_{10} \), would vanish if one chooses the origin to coincide with the center of figure. The fact that they are non-zero reflects the offset in the location of the origin used to measure the shape control points of Duxbury and
Callahan [1989], which are not evenly distributed on the surface. In addition, there is an offset of the center of mass from the center of figure because the two centers generally do not coincide. The dipole gravitational coefficients, $C_{20}$ and $S_{20}$, are non-zero as well. It is, however, customary (and desirable) to express the gravitational coefficients in the center-of-mass coordinate system. Thus an empirical translation of origin ($Ax = 30$ m, $Ay = -279$ m, $Az = -103$ m) is applied (analytically) to the gravitational coefficients. The ensuing values, both normalized and un-normalized, are presented in Table 1.

Apart from the small libration and possibly a wobble whose effect averages out over time, the coordinate axes (as defined and with origin at the center of mass) should coincide with Phobos' principal axes. Consequently, the coefficients $C_{11}$, $S_{11}$, and $S_{20}$, being proportional respectively to the (off-diagonal) $ex$, $yz$, and $xy$ components of the inertia tensor [Chao and Gross, 1987, p.574], should vanish. The fact that the computed values do not vanish reflects the slight difference of the shape model of Duxbury and Callahan [1989] from Phobos' true shape. The computed axis of the least moment of inertia is misaligned from the $x$ axis by $[\tan^{-1}(S_{22}/C_{22}) / 2] \approx 0.26^\circ$ [cf. Chao and Gross, 1987].

As far as gravitational coefficients are concerned, the above-mentioned uncertainties in coordinate origin and axis orientation should be considered as sources of error. They are on the order of a few percent, and can be reduced if improved shape models are used. An improvement over the present shape model can be achieved through a finer control network and, perhaps more importantly, the inclusion of crater topography.

### Table 1. Normalized gravitational harmonic coefficients for Phobos (in units of $10^{-2}$), with reference to the equivalent-volume radius in the center-of-mass coordinate system. Corresponding un-normalized values are given in parentheses; and the zonal $J_2$ coefficient is equal to the negative of the un-normalized $C_{10}$.

<table>
<thead>
<tr>
<th>$lm$</th>
<th>$C_{lm}$</th>
<th>$S_{lm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-4.698</td>
<td>0.138</td>
</tr>
<tr>
<td>21</td>
<td>0.136</td>
<td>0.178</td>
</tr>
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<td>22</td>
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<td>30</td>
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<td>0.181</td>
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<td>31</td>
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</tr>
<tr>
<td>32</td>
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<td>-0.0655</td>
</tr>
<tr>
<td>33</td>
<td>0.224</td>
<td>-1.392</td>
</tr>
<tr>
<td>40</td>
<td>0.762</td>
<td>0.0115</td>
</tr>
<tr>
<td>41</td>
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</tr>
<tr>
<td>42</td>
<td>-0.0288</td>
<td>-0.112</td>
</tr>
<tr>
<td>43</td>
<td>-0.280</td>
<td>0.337</td>
</tr>
<tr>
<td>44</td>
<td>-0.120</td>
<td>-0.0622</td>
</tr>
</tbody>
</table>

especially for the dominant crater Stickney which is invisible in Duxbury's [1989] topography as he points out. In relative terms, any improvement in the shape model will result in a similar improvement in the gravitational field: the factor $R(\Omega)^{1/3}$ in integral (6) for the gravitational coefficient magnifies any error in the shape $R(\Omega)$ by a factor of $1+\delta$, but only to be largely offset by the $(21+1)$-fold decrease in the first-order term (equation 9). Thus the uncertainty in Table 1 is believed to be a few percent.

We are now in a position to discuss the in-plane (longitudinal) physical libration in Phobos' synchronous rotation. This libration has a large amplitude [e.g., Veverka and Burns, 1980] because of a near resonance between the orbital period, $P_0$ (0.3189 days for Phobos), and the free libration period, $(3\pi)^{-1/2} P_0$. The parameter

$$\gamma = (B - A) / C$$

where $A < B < C$ are the three principal moments of inertia. The predicted amplitude is

$$\theta = 2\epsilon / [1 - (1 / 3\gamma)]$$

[e.g., Peale, 1977], where $\epsilon$ is the orbit eccentricity (0.015 for Phobos).

For simplicity we shall express $A$, $B$, and $C$ in units of $M_0R_0^2$. Given the shape and the uniform density, they can be computed by direct integration. However, a simpler and more enlightening alternative is to utilize the gravitational coefficients. To do this one needs an extra inertia tensor parameter. This is because the inertia tensor has six elements while there are only five quadrupole ($l=2$) gravitational coefficients that relate to the inertia tensor; a fact pertaining to the classic non-uniqueness in the gravitational inverse problem [Horse & Feshbach 1953, p. 1276-1283]. One such parameter which is particularly simple is the trace $T = A + B + C$; and it can be shown that

$$B - A = 2\sqrt{5} \ C_{22} / \sqrt{3}$$

and

$$C = (T - 2\sqrt{5} \ C_{20}) / 3$$

in the principal axes [for details see e.g. Chao and Gross, 1987, p.473]. A numerical integration yields $T = 1.258$ (compared with 1.2 for a uniform sphere, which is the lower bound for a homogeneous body). Given the values of $C_{20}$ and $C_{22}$ in Table 1, we then find

$$A = 0.355, \quad B = 0.414, \quad C = 0.489$$

(compared with 0.4 for a uniform sphere). Substituting equation (13) into (10), we get $\gamma = 0.120$. Our predicted in-plane libration amplitude (equation 11) for Phobos is therefore $\theta = 0.97^\circ$, within the rather large standard deviation of the observed value $0.78 \pm 0.4^\circ$ reported by Duxbury [1989] using Viking Mission data. Duxbury gave, in addition, a predicted value of 0.81°. This unfortunately is based on gravitational coefficients (especially $C_{22}$) resulting from equation (9) which we have shown to be inadequate.

It should be pointed out here that $B$ is rather sensitive to the density distribution. This is because of the closeness of $A$ to $B$ in equation...
(10): a small error in $A$ and/or $B$ (due to any possible departure of the true density from a uniform distribution) can give rise to a large error in $\eta$, and consequently in $B$. For example, if the true $A$ is larger than equation (13) by 5% and $B$ smaller by 5%, then the resultant $B$ becomes as small as 0.25°. Therefore, unless fortuitous, the loose agreement between the predicted and observed $\theta$ is indicative of a uniform density for Phobos. In any event, a more accurate observation of $\theta$, when compared with the predicted value (preferably based on some improved shape models), will provide useful constraints on the density distribution of Phobos.

There are other, but probably less effective dynamical tests for the assumption of a homogeneous Phobos. The most direct is the orbital motion of spacecraft near Phobos, from which the low-degree gravitational field of Phobos can be inferred. This field also has long-term influences on Phobos' orbit around Mars. Chapront-Touze (1988) has considered Phobos' $C_{20}$ and $C_{22}$ and found that, in addition to the main effect due to Mars' anomalous gravitational field, they contribute to the precession of the node and pericenter of Phobos' orbit. Unfortunately, her values (2 x 10^{-5} day^{-1} for the node and 1 x 10^{-3} day^{-1} for the precession of the longitude of pericenter) are based on Barkin's (1982) gravitational coefficients which, in turn, are based on the values of Sagitov et al. (1981) (see Section 1). Our predicted values according to her equations are an order of magnitude smaller: 3.4 x 10^{-7} day^{-1} for the node and -9.7 x 10^{-5} day^{-1} for the pericenter. Additional effects due to the third and higher harmonics can be shown to be insignificant [Eckhardt, 1973; Barkin, 1985].

The acquisition of better observations has, of course, to await future expeditions, an example being the (unsuccessful) Soviet PHOBOS Mission which had planned close encounters and landing of spacecraft on Phobos [Sagitov, 1988]. Other methods for learning Phobos' density distribution, such as seismic methods and possibly drilling, have also been under consideration. In fact, Phobos has been designated as an outpost for human expedition to Mars in the NASA 'Pathway to Exploration' concept studies [NASA, 1988]. These expeditions can provide as well as benefit from a knowledge of Phobos' gravitational field.

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